An Exploration of the Cantor Set

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James and Elizabeth Monahan Summer Research Fellowship – Summer 2008

MT4960: Mathematics Seminar – Spring 2009

Abstract

This paper is a summary of some interesting properties of the Cantor ternary set and a few investigations of other general Cantor sets. The ternary set is discussed in detail, followed by an explanation of three ways of creating general Cantor sets developed by the author. The focus is on the dimension of these sets, with a detailed explaination of Hausdorff dimension included, and how they act as interesting examples of fractal sets.

Keywords: Cantor set, fractal, Hausdorff dimension, self-similarity

Introduction

Georg Cantor (1845-1918) first introduced the set that became known as the Cantor ternary set in the footnote to a statement saying that perfect sets do not need to be everywhere dense. This footnote gave an example of an infinte, perfect set that is not everywhere dense in any interval, a set he defined as real numbers of the form

$$x = \frac{c_1}{3} + ... + \frac{c_v}{3^v} + ... \tag{1.1}$$

where c_v is 0 or 2 for each integer v.[1]

Definition (1.2): A set S is said to be perfect if S = S', where S' is the set of all the limit points of S. In other words, S is perfect if S is a closed set in which every point is a limit point of S.

Definition (1.3): A set S is said to be nowhere dense if the interior of the closure of S is empty.

The paper in which Cantor's statement and footnote were made was written in October of 1882, but in a paper published in 1875, H. J. S. Smith made the discovery of nowhere dense sets with positive outer content, meaning that they still take up space. This paper, in which Smith gave an example of a set that would today be classified as a type of Cantor set, went largely unnoticed until well after Cantor's discoveries were made. Now, in this paper, I attempt to study the Cantor ternary set from the perspective of fractals and Hausdorff dimension and make my own investigations of other general Cantor sets I constructed and their dimensions.

The term "Cantor set" is most often used to refer to what is known as the Cantor ternary set, which is constructed as follows:

Let I be the interval $[0, 1]$. Divide I into thirds. Remove the open set that makes up the middle third, that is, $(\frac{1}{3}, \frac{2}{3})$, and let A_1 be the remaining set. Then

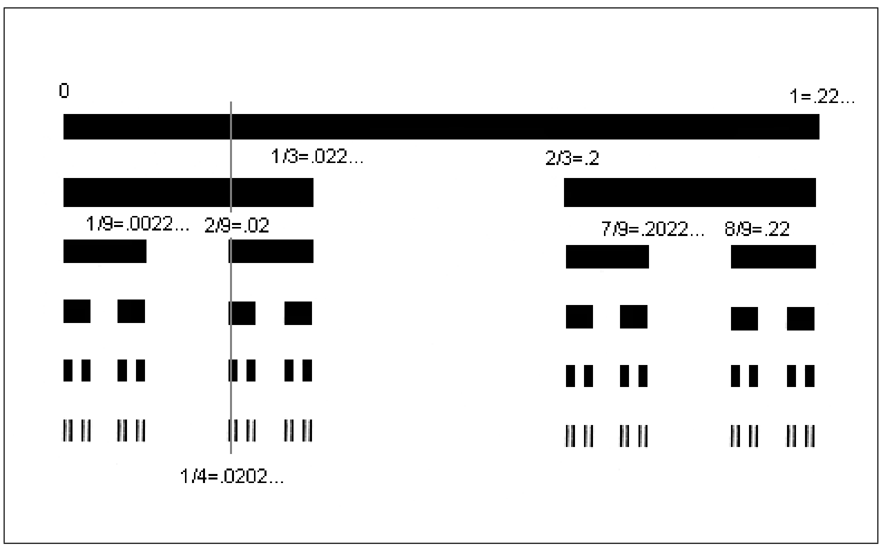
$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]. \tag{1.4}$$

Continue by removing the open middle third segment from each of the two closed sets in A_1 and call the remaining set A_2. So,

$$A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]. \tag{1.5}$$

Continue in this fashion at each step k for $k \in \mathbb{N}$, removing the open middle third segment from each of the closed sets in A_k and calling the remaining set A_{k+1}. For each $k \in \mathbb{N}$, A_k is the union of 2^k closed intervals each of length 3^{-k}.

Definition (1.6): The Cantor ternary set, which we denote C_3, is the "limiting set" of this process, i.e. $C_3 = \bigcap_{k=1}^{\infty} A_k$. [2]



The Cantor ternary set is interesting in mathematics because of the apparent paradoxes of it. By the way it is constructed, an infinite number of intervals whose total length is 1 are removed from an interval of length 1, so the set cannot contain any interval of non-zero length. Yet the set does contain an infinite number of points, and, in fact, it has the cardinality of the full interval $[0, 1]$. So the Cantor set contains as many points as the set it is carved out of, but contains no intervals and is nowhere dense. We know that the set contains an infinite number of points because the endpoints of each closed interval will always remain in the set, but the Cantor set actually contains more than just the endpoints of the closed intervals A_k. In fact,

$\frac{1}{4} \in C$ but is not an endpoint of any of the intervals in any of the sets A_k. Notice first that each element of C can be written in a ternary (base 3) expansion of only 0s and 2s. At every level of removal, every number with a ternary expansion involving a 1 is removed. At the first stage of removal, for instance, any number remaining would have the digit $c_1 = 0$ or 2 where $x = 0.c_1c_2c_3...$, since if $x \in [0, \frac{1}{3}]$, $c_1 = 0$ and if $x \in [\frac{2}{3}, 1]$, $c_1 = 2$. Repeating this argument for each level of removal, it can be shown that if x remains after removal n, c_n is 0 or 2. Now, we can write $\frac{1}{4}$ as $0.0202\overline{02}...$ in ternary expansion. At the k^{th} stage of removal, any new endpoint has a form of either a 2 in the $3^{-k^{th}}$ ternary place which repeats infinitely or terminates at the $3^{-(k-1)}$ ternary place. Since $\frac{1}{4} = 0.0202\overline{02}$, it does not follow the pattern of the endpoints. Therefore, $\frac{1}{4}$ is not an endpoint of the Cantor set, and there are infinitely many points like that.

Properties of the Cantor set

Certain easily proven properties of the Cantor ternary set, when they are pieced together, help to show the special nature of Cantor sets. The Cantor ternary set, and all general Cantor sets, have uncountably many elements, contain no intervals, and are compact, perfect, and nowhere dense.

C_3 has uncountably many elements.

We will show this by contradiction. Let $C_3 = \{x \in [0, 1) : x$ has a ternary expansion involving only zeros and twos$\}$. Suppose C_3 is countable. Then there exists $f : \mathbb{N} \to^{1-1}_{onto} C_3$ by the definition of countablility. Let $X_n = f(n)$ for all $n \in \mathbb{N}$. So $C_3 = \{x_1, x_2, x_3, ..., x_n, ...\}$ where:

$$x_1 = 0.c_{1_1}c_{1_2}c_{1_3}...$$

$$x_2 = 0.c_{2_1}c_{2_2}c_{2_3}...$$

$$\vdots \tag{2.1}$$

$$x_n = 0.c_{n_1}c_{n_2}c_{n_3}...$$

$$\vdots$$

where c_{n_m} is either 0 or 2 for all n, m. Define $c = 0.c_1c_2c_3...$ by

$$c_1 = \begin{Bmatrix} 2 \text{ if } c_{1_1} = 0 \\ 0 \text{ if } c_{1_1} = 2 \end{Bmatrix}, c_2 = \begin{Bmatrix} 2 \text{ if } c_{2_2} = 0 \\ 0 \text{ if } c_{2_2} = 2 \end{Bmatrix}, ..., c_n = \begin{Bmatrix} 2 \text{ if } c_{n_n} = 0 \\ 0 \text{ if } c_{n_n} = 2 \end{Bmatrix}, ... \tag{2.2}$$

Clearly, $c \in C_3$. But, $c \neq X_n$ for any n, since $c \neq x_n$ in its $3^{-n^{th}}$ place. This is a contradiction. Therefore, C_3 is uncountable.

C_3 contains no intervals.

We will show that the length of the complement of the Cantor set C_3 is equal to 1, hence C_3 contains no intervals. At the k^{th} stage, we are removing 2^{k-1} intervals from the previous set of intervals, and each one has a length of $\frac{1}{3^k}$. The length of the complement within $[0, 1]$ after an infinite number of removals is:

$$\sum_{k=1}^{\infty} 2^{k-1}(\frac{1}{3^k}) = \frac{1}{3}\sum_{k=1}^{\infty}(\frac{2}{3})^{k-1} = \frac{1}{3}\sum_{k=0}^{\infty}(\frac{2}{3})^k = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1. \tag{2.3}$$

Thus, we are removing a length of 1 from the unit interval $[0, 1]$ which has a length of 1. Therefore, the Cantor set must have a length of 0, which means it has no intervals.

C_3 is compact.

Using the Heine-Borel Theorem, which states that a subset of $\mathbb{R}$ is compact iff it is closed and bounded, it can be shown rather easily that C_3 is compact. C_3 is the intersection of a collection of closed sets, so C_3 itself is closed. Since each A_k is within the interval $[0, 1]$, C_3, as the intersection of the sets A_k, is bounded. Hence, since C_3 is closed and bounded, C_3 is compact.

So far we have that the Cantor set is a subset of the interval $[0, 1]$ that has uncountably many elements yet contains no intervals. It has the cardinality of the real numbers, yet it has zero length.

C_3 is perfect.

By (1.2) a set is perfect if the set is closed and all the points of the set are limit points of the set. Since C_3 is compact, it is necessarily closed. For each endpoint in the set C_3 there will always exist another point in the set within a deleted neighborhood of some radius $\varepsilon > 0$ on one side of that point since the remaining intervals at each step are being divided into infinitely small subintervals and since the real numbers are infinitely dense. Likewise, for each nonendpoint in the set there will always exist another point in the set within a deleted neighborhood of some radius $\varepsilon > 0$ on both sides of that point. Hence, there must exist a deleted neighborhood of some radius $\varepsilon > 0$ around each point of the set C_3 for which the intersection of that deleted neighborhood and the set is nonempty. Therefore, each point in the set is a limit point of the set, and since the set is closed, the set C_3 is perfect.

C_3 is nowhere dense.

By (1.3) a set is nowhere dense if the interior of the closure of the set is empty. The closure of a set is the union of the set with the set of its limit points, so since every point in the set C_3 is a limit point of the set the closure of C_3 is simply the set itself. Now, the interior of the set must be empty since no two points in the set are adjacent to each other. At the infinite level of removal, if there did exist a series of adjacent points, that is an interval of points, the middle third section of that interval will be removed and the removal would continue on an infinitely small scale, ultimately removing anything between two points. Hence the set C_3 is nowhere dense.

Dimension of the Cantor Set

The Cantor set seems to be merely a collection of non-adjacent points, so should intuitively have a dimension of zero, as any random collection of non-adjacent points would have. In this sense, the topological dimension of the Cantor set is 0. But utilizing a different definition of dimension, such as Hausdorff dimension, allows us to see the fractional dimension of the Cantor set while still maintaining the integer dimensions of points, lines, and planes.

Hausdorff Dimension

Definition: Let $E \subseteq \mathbb{R}^n$ and let s and δ be positive real numbers. Let $C_\delta(E)$ be the collection of all countable δ-covers of E, where a δ-cover of E is a sequence $\{U_j\}$ of subsets of $\mathbb{R}^n$ whose union contains E and for which $0 < |U_j| < \delta$ for all $j = 1, 2, ...$ Then,

$$H^s_\delta(E) := \inf\{\sum_{j=1}^{\infty} |U_j|^s : \{U_j\} \in C_\delta(E)\} \tag{3.1}$$

[3].

Definition (3.2): For any subset E of $\mathbb{R}^n$, $H^s(E) := \sup_{\delta \to 0} H^s_\delta(E)$.

Definition (3.3): The Hausdorff s-dimensional measure is the restriction of $H^s(*)$ to the σ-field of $H^s(*)$-measureable sets. The Hausdorff dimension of E is defined to be:

$$\dim(E) = \inf\{s : H^s(E) = 0\} \tag{3.4}$$

[3].

In simple terms, Hausdorff dimension involves first considering the δ-covers of the set E. If you take the sum of the diameters of all δ-covers of E raised to the power of s, the infimum of that sum is $H^s_\delta(E)$. The Hausdorff s-dimensional measure of E is then the $\sup_{\delta > 0} H^s_\delta(E)$. Finally, the Hausdorff dimension of E, denoted by $\dim(E)$, is the infimum of all the values of the real number s such that the Hausdorff s-dimensional measure of E is 0.

Hausdorff dimension generalizes the concept of dimension of a vector space in such a way that points have Hausdorff dimension 0, lines have Hausdorff dimension 1, etc., but in general the Hausdorff dimension of a set is not necessarily integer. Fractals are defined as sets whose Hausdorff dimension is greater than its topological dimension, with the Hausdorff dimension of fractals specifically non-integer.

Theorem 1: The dimension of the Cantor ternary set (C_k) is:

$$d = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})} \tag{3.5}$$

Proof: Let us create a series of δ-covers of C_k:

$$\begin{aligned}
\{U_{0_1} &= [0,1]\}, \text{ a 1 cover} & (3.6)\\
\{U_{1_1} &= [0, \frac{1}{2} - \frac{1}{2k}], U_{1_2} = [\frac{1}{2} + \frac{1}{2k}, 1]\}, \text{ a } (\frac{1}{2} - \frac{1}{2k}) \text{ cover} \\
\{U_{2_1} &= [0, \frac{1}{4} - \frac{1}{2k} + \frac{1}{4k^2}], U_{2_2} = [\frac{1}{4} - \frac{1}{4k^2}, \frac{1}{2} - \frac{1}{2k}], U_{2_3} = [\frac{1}{2} + \frac{1}{2k}, \frac{3}{4} + \frac{1}{4k^2}], \\
U_{2_4} &= [\frac{3}{4} + \frac{1}{2k} - \frac{1}{4k^2}, 1]\}, \text{ a } (\frac{1}{4} - \frac{1}{2k} + \frac{1}{4k^2}) \text{ cover} & (1)
\end{aligned}$$

In general, a $(\frac{1}{2} - \frac{1}{2k})^g$-cover is

$$\{U_{g_j}\}_{j=1}^{2^g} \tag{3.7}$$

For $s > 0$,

$$\begin{aligned}
H^s_\delta(C_k) \quad : &= \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(C_k)\} & (3.8)\\
&\leq \sum_{i=1}^{2^g} |U_{gi}|^s \text{ , for } (\frac{1}{2} - \frac{1}{2k})^g \leq \delta \\
&= 2^g((\frac{1}{2} - \frac{1}{2k})^g)^s = \frac{2^g}{((\frac{1}{2} - \frac{1}{2k})^g)^s} = \left(\frac{2}{(\frac{1}{2} - \frac{1}{2k})^s}\right)^g
\end{aligned}$$

So, if $(\frac{1}{2} - \frac{1}{2k})^s = 2$, then $H^s_\delta(C_k) \leq 1$ for all $\delta > 0$. Now, $(\frac{1}{2} - \frac{1}{2k})^s = 2$ iff $\log(\frac{1}{2} - \frac{1}{2k})^s = \log 2$ iff $s \log(\frac{1}{2} - \frac{1}{2k}) = \log 2$. That is, $s = \frac{\log 2}{\log(\frac{1}{2} - \frac{1}{2k})}$. If $s = \frac{\log 2}{\log(\frac{1}{2} - \frac{1}{2k})}$, then $H^s(C_k) \leq 1$ and

$$\dim(C_k) \leq \frac{\log 2}{\log(\frac{1}{2} - \frac{1}{2k})} \tag{3.9}$$

Now we show $H^s(C_k) \geq 1$, where $s = \frac{\log 2}{\log(\frac{1}{2} - \frac{1}{2k})}$. For $\delta > 0$, let $\{V_i\}$ be a δ-cover of C_k. We may assume V_i is open for each i. Since C_k is compact, there is a finite subcover $\{V_i\}_{i=1}^n$; $V_i = (a_i, b_i)$, where $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for $1 \leq i \leq n-1$. Now, there exists a closed subinterval $[\alpha_i, \beta_i]$ of (a_i, b_i) such that

$$C_k \subseteq \bigcup_{i=1}^{n} [\alpha_i, \beta_i], \alpha_i < \beta_i < \alpha_{i+1} \leq \beta_{i+1} \text{ for } 1 \leq i \leq n-1; \tag{3.10}$$

where α_i and β_i have the form $m(\frac{1}{2} - \frac{1}{2k})^g$ for all i.

Let

$$K = \max\{k : \alpha_i = m(\frac{1}{2} - \frac{1}{2k})^g \text{ or } \beta_i = m(\frac{1}{2} - \frac{1}{2k})^g \text{ for some } i, 1 \leq i \leq n;\ (\frac{1}{2} - \frac{1}{2k}) \text{ doesn't divide } m\} \tag{3.11}$$

Then $\bigcup_{j=1}^{2^g} U_{g_j} \subseteq \bigcup_{i=1}^{n} [\alpha_i, \beta_i]$, where $\{U_{g_j}\}_{j=1}^{2^g}$ is as defined in the first part of the proof.

Let $C_g = \bigcup_{j=1}^{2^g} U_{g_j}$. Now consider $[\alpha_i, \beta_i] \setminus C_g$ for some i, $1 \leq i \leq n$. $[\alpha_i, \beta_i] \setminus C_g$ contains at least one open interval, I, of length at least $\frac{1}{(\frac{1}{2} - \frac{1}{2k})}(\beta_i - \alpha_i)$. Now $[\alpha_i, \beta_i] \setminus I$ is the disjoint union of two closed intervals, I_1 and I_2. By the concavity of $f(t) = t^s$ and since $(\frac{1}{2} - \frac{1}{2k})^s = 2$ and $(\beta_i - \alpha_i) \geq \frac{(\frac{1}{2} - \frac{1}{2k})}{2}(|I_1| + |I_2|)$ we obtain:

$$\begin{aligned}
(\beta_i - \alpha_i)^s &\geq [\frac{(\frac{1}{2} - \frac{1}{2k})}{2}(|I_1| + |I_2|)]^s \\
&= 2[\frac{1}{2}|I_1| + \frac{1}{2}|I_2|]^s \\
&\geq 2(\frac{1}{2}|I_1|^s + \frac{1}{2}|I_2|^s) \\
&= |I_1|^s + |I_2|^s
\end{aligned} \tag{3.12}$$

Reduce I_1 and I_2 similarly and obtain that $(\beta_i - \alpha_i)^s \geq \sum_{j: U_{gi} \subseteq [\alpha_i, \beta_i]} |U_{gi}|^s$

Consequently,

$$\begin{aligned}
\sum_{i=1}^{n}|V_i|^s &\geq \sum_{i=1}^{n}(\beta_i-\alpha_i)^s \qquad (3.13)\\
&\geq \sum_{i=1}^{n}\left(\sum_{j:U_{g_j}\subseteq[\alpha_i,\beta_i]}|U_{gi}|^s\right)\\
&= \sum_{j=1}^{2^g}|U_{gj}|^s\\
&= 2^g\left((\frac{1}{2}-\frac{1}{2k})^g\right)^s = \left(\frac{2}{(\frac{1}{2}-\frac{1}{2k})^s}\right)^g = 1
\end{aligned}$$

It follows that $H_\delta^s(C_k)\geq 1$ and hence $H^s(C_k)\geq 1$. Therefore, $\dim(C_k)=\frac{\log 2}{\log(\frac{1}{2}-\frac{1}{2k})}$.■ [**3**]

There is a shortcut for computing the Hausdorff dimension of sets that are self-similar in nature. To be self-similar, the mappings of the set must be a finite collection of similitudes, that is, the mappings preserve the geometry of the set, such that the set is invariant with respect to the set of mappings, and there must exist a positive real number s such that H^s of the set is positive but H^s of the intersection of two different mappings of the set is zero.

Definition (3.14): Let $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$. If there exists a c, $0<c<1$, such that $|\phi(x)-\phi(y)|\leq c\,|x-y|$ for any x and y in $\mathbb{R}^n$, then ϕ is called a contraction.

Definition (3.15): Let ϕ be a contraction. Then $\inf\{c:|\phi(x)-\phi(y)|\leq c\,|x-y|\,\forall x,y\epsilon\mathbb{R}^n\}$ is called the ratio of the contraction ϕ.

Definition (3.16): Let $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be a contraction and E be a subset of $\mathbb{R}^n$. If ϕ preserves the geometry of E (ϕ is a combination of a translation, a rotation, a reflection, and/or a dilation) then ϕ is called a similitude.

Definition (3.17): Let $E\subseteq\mathbb{R}^n$ and $\{\phi_i\}_{i=1}^k$ be a finite collection of contractions. Then E is said to be invariant with respect to $\{\phi_i\}_{i=1}^k$ if $E=\bigcup_{i=1}^{k}\phi_i(E)$.

Definition (3.18): Let $E\subseteq\mathbb{R}^n$ and $\{\phi_i\}_{i=1}^k$ be a finite collection of similitudes such that E is invariant with respect to $\{\phi_i\}_{i=1}^k$. If there exists $s>0$ such that $H^s(E)>0$ but $H^s(\phi_i(E)\cap\phi_j(E))=0$ for $i\neq j$, then E is called self-similar.

Definition (3.19): A finite collection of contractions $\{\phi_i\}_{i=1}^k$ is said to have the open set condition if there exists a bounded open set W such that $\bigcup_{i=1}^{k} \phi_i(W) \subseteq W$, and this union is disjoint.

The Cantor set is self-similar by (3.18) since its mappings, all variations on $\frac{1}{3}x$, preserve the geometry of the set and there is a positive Hausdorff s-dimensional measure of the set but not of the intersection of two different mappings since no two mappings of the set have a non-empty intersection. The Hausdorff dimension of a self-similar set can be found by using the following theorem:

Theorem 2: Let $\{\Phi_i\}_{i=1}^k$ be a collection of similitudes such that $E \subseteq \mathbb{R}^n$ is invariant with respect to $\{\Phi_i\}_{i=1}^k$. If $\{\Phi_i\}_{i=1}^k$ satisfies the open set condition and r_i is the ratio of the i-th similitude Φ_i, then the Hausdorff dimension of E is equal to the unique positive numbers for which $\sum_{i=1}^{k}(r_i)^s = 1$. [**3**]

The computation of the Hausdorff dimension of the Cantor ternary set, C_3, follows very easily from Theorem 2.

Proposition: The dimension of the Cantor ternary set C_3 is $d = \frac{\log 2}{\log 3}$.

Proof: Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\phi_1(x) = \frac{1}{3}x$$

$$\phi_2(x) = \frac{1}{3}x + \frac{2}{3} \tag{3.20}$$

Notice that $C_3 = \bigcup_{i=1}^{2} \Phi_i(C_3)$. Also, $\{\Phi_i\}_{i=1}^2$ satisfies the open set condition for $W = (0, 1)$. Applying the theorem with $r_1 = \frac{1}{3}$ and $r_2 = \frac{1}{3}$, we need to find s such that

$$\sum_{i=1}^{2}(r_i)^s = 1.$$

$$2(\frac{1}{3})^s = 1 \text{ iff } s = \frac{\log 2}{\log 3}. \tag{3.21}$$

$$\dim(C_3) = \frac{\log 2}{\log 3}.\blacksquare$$

[3].

General Cantor Sets

Up to this point, our discussion of the Cantor set has been limited to what is known as the Cantor ternary set, defined in the Introduction. We will now discuss some generalizations. My further investigations were motivated by a curiosity as to what would happen to the dimension of the set if the removal process was defined differently. I consider three different general methods of removal from the interval $[0, 1]$ that depend upon a natural number k. As will be shown, all three methods of removal are equivalent when $k = 3$, yielding the Cantor ternary set.

1.) Method C: Let $\{C_k\}$ be the collection of sets defined in terms of k, for $k \geq 2$, in which each set in the sequence is formed by the repetitive removal of an open interval of length $\frac{1}{k}$ from the center of each closed interval, starting with the interval $[0, 1]$. In this way, the size of the closed intervals remaining on either side of the open interval removed will be $(\frac{1}{2} - \frac{1}{2k})$.

Since each of the sets C_k are self-similar sets, we can use Theorem 2 to find the Hausdorff dimension of each of the sets, which we will do in general for any $k \geq 2$.

Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\phi_1(x) = (\frac{1}{2} - \frac{1}{2k})x$$

$$\phi_2(x) = (\frac{1}{2} - \frac{1}{2k})x + \frac{1}{2} + \frac{1}{2k} \tag{4.1}$$

Notice that $C_k = \bigcup_{i=1}^{2} \phi_i(C_k)$. Applying the theorem for self-similar sets with $r_1 = (\frac{1}{2} - \frac{1}{2k})$ and $r_2 = (\frac{1}{2} - \frac{1}{2k})$, we need to find s such that

$$\sum_{i=1}^{2} (r_i)^s = 1. \tag{4.2}$$

So,

$$2(\frac{1}{2} - \frac{1}{2k})^s = 1 \text{ iff } s = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}. \tag{4.3}$$

Thus,

$$\dim(C_k) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}. \blacksquare \tag{4.4}$$

2.) Method D: Let $\{D_k\}$ be the collection of sets defined in terms of k, for $k \geq 2$, in which each set in the sequence is formed by the repetitive removal of an open interval of length $(1 - \frac{2}{k})$ from the center of each closed interval starting with $[0, 1]$, with intervals of length $\frac{1}{k}$ remaining on each side. Notice that with this method of removal, we are varying the length of the side intervals in terms of k then removing the interval inbetween, whereas in the first method of removal, method C, you are varying the length of the center interval in terms of k.

Again, since each of the sets D_k are self-similar sets, we can use Theorem 2 to find the Hausdorff dimension of each of the sets, which we will do in general for any $k \geq 2$.

Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\begin{aligned} \phi_1(x) &= \frac{1}{k}x \\ \phi_2(x) &= \frac{1}{k}x + 1 - \frac{1}{k} \end{aligned} \tag{4.5}$$

Notice that $D_k = \bigcup_{i=1}^{2} \phi_i(D_k)$. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$ and $r_2 = \frac{1}{k}$, we need to find s such that

$$\sum_{i=1}^{2} (r_i)^s = 1. \tag{4.2}$$

So,

$$2(\frac{1}{k})^s = 1 \text{ iff } s = \frac{\log 2}{\log k}. \tag{4.6}$$

Thus,

$$\dim(D_k) = \frac{\log 2}{\log k}. \blacksquare \tag{4.7}$$

3.) Method E: Let $\{E_k\}$ be the collection of sets defined in terms of k, for $k \geq 2$, in which each set in the sequence is formed by the repetitive removal of alternating open intervals of length $\frac{1}{k}$ from each closed interval, starting with $[0, 1]$, when each closed interval is divided into k subintervals. This method of removal leads to two similar but distinctly different cases. When k is odd, $\frac{k-1}{2}$ alternating sections are removed from each closed interval, leaving each interval of length $\frac{1}{k}$. When k is even, $(\frac{k}{2} - 1)$ alternating sections are removed from each closed interval, leaving one interval of length $\frac{1}{k}$ on the left end and two full intervals each of length $\frac{1}{k}$ adjacent to each other on the right end.

Since different mappings will be required in order to generate the sets, each of the two cases yields sets with different Hausdorff dimensions. In the case where k is odd, using Theorem 2 to find the Hausdorff dimension of these self-similar sets, we let $\phi_1(x)$, $\phi_2(x)$, ... , $\phi_{\frac{k+1}{2}}(x)$ be defined as:

$$\begin{aligned} \phi_1(x) &= \frac{1}{k}x \\ \phi_2(x) &= \frac{1}{k}x + \frac{2}{k} \\ &, ..., \\ \phi_{\frac{k+1}{2}}(x) &= \frac{1}{k}x + \frac{k-1}{k} \end{aligned} \tag{4.8}$$

Note that the number of ϕ_i mappings needed is determined by the value of the natural number k, with $\frac{k+1}{2}$ mappings needed when k is odd. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$, $r_2 = \frac{1}{k}$, ... , $r_{\frac{k+1}{2}} = \frac{1}{k}$, we need to find s such that

$$\sum_{i=1}^{\frac{k+1}{2}} (r_i)^s = 1. \tag{4.2}$$

So,

$$(\frac{k+1}{2})(\frac{1}{k})^s = 1 \text{ iff } s = \frac{\log \frac{k+1}{2}}{\log k}. \tag{4.9}$$

Thus, when k is odd,

$$\dim(E_k) = \frac{\log \frac{k+1}{2}}{\log k}. \blacksquare \tag{4.10}$$

In the case where k is even, we let $\phi_1(x)$, $\phi_2(x)$, ... , $\phi_{\frac{k+1}{2}}(x)$ be defined as:

$$\begin{aligned} \phi_1(x) &= \frac{1}{k}x \\ \phi_\alpha(x) &= \frac{1}{k}x + \frac{2}{k} \\ &,..., \\ \phi_{\frac{k}{2}}(x) &= \frac{2}{k}x + \frac{k-2}{k} \end{aligned} \tag{4.11}$$

for $\alpha = 1, 2, ..., (\frac{k}{2} - 1)$. Note that the number of ϕ_i mappings is determined by the natural number k, with $\frac{k}{2}$ mappings needed when k is even. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$, $r_\alpha = \frac{1}{k}$, $r_{\frac{k}{2}} = \frac{2}{k}$, we need to find s such that

$$\sum_{i=1}^{\frac{k}{2}} (r_i)^s = 1. \tag{4.2}$$

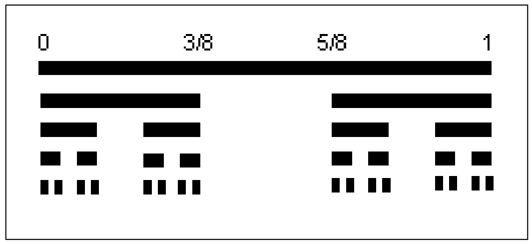
So,

$$(\frac{k}{2} - 1)(\frac{1}{k})^s + (\frac{2}{k})^s = 1. \tag{4.12}$$

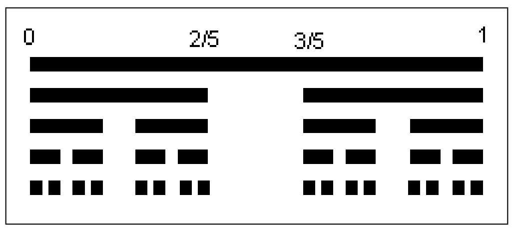
Simplifying this equation to $(\frac{k}{2} - 1) = k^s - 2^s$, it is not noticeably solvable directly for a general k, but the dimension of each set is equal to the value of s that satisfies the equation for its given k.

The differences between the sets formed under each method of removal are apparent when $k = 4$, and the differences between different values of the natural number k for a given method of removal are apparent by comparing $k = 4$ with $k = 5$.

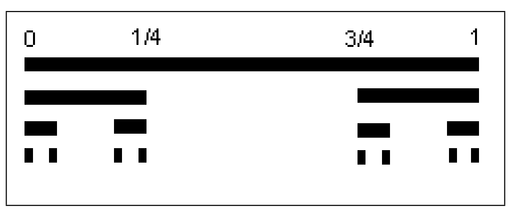
Set C_4:



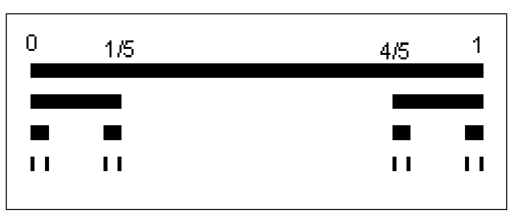
Set C_5:



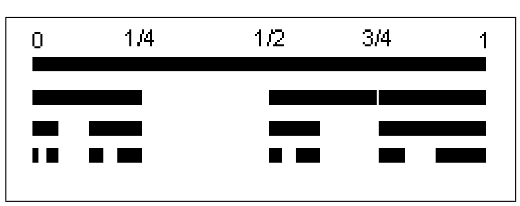
Set D_4:



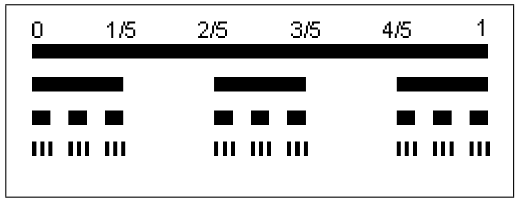
Set D_5:



Set E_4:



Set E_5:



Note that when $k = 2$, the amount being removed under methods D and E is equal to 0, and the dimension is consequently equal to 1 in all cases, since we are left with the full interval $[0, 1]$.

Note**:** $C_3 = D_3 = E_3$.

Proof: The set C_3 is formed by the repetitive removal of an open interval of length $\frac{1}{3}$ from the center of each closed interval, starting with the interval $[0, 1]$ which leaves closed intervals of size $\frac{2}{3}$ on each side. Since the set D_3 is formed by the repetitive removal of an open interval of length $(1 - \frac{2}{3}) = \frac{1}{3}$ from the center of each closed interval starting with $[0, 1]$, with intervals of length $\frac{1}{3}$ remaining on each side, it is equivalent to the set C_3. Also, since the set E_3 is formed by dividing the interval $[0, 1]$ into 3 subintervals and removing alternating sections, which is only the center section, with intervals of length $\frac{1}{3}$ on each end, it is equivalent to both C_3 and D_3. Hence, $C_3 = D_3 = E_3$.$\blacksquare$

Also note that the calculations of dimension for each set yield the same dimension, which must be the case since the three sets are the same. So,

$$\dim(C_3) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{6})} = \frac{\log 2}{\log 3}. \quad \dim(D_3) = \frac{\log 2}{\log 3}. \quad \dim(E_3) = \frac{\log \frac{3+1}{2}}{\log 3} = \frac{\log 2}{\log 3} \tag{4.13}$$

Thus, $\dim(C_3) = \dim(D_3) = \dim(E_3)$.

In each method of removal, what happens to the dimension as k approaches infinity?

In method C, $\dim(C_k) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}$. So,

$$\lim_{k\to\infty} \dim(C_k) = \lim_{k\to\infty} \left(\frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})} \right) = \frac{\log \frac{1}{2}}{\log \frac{1}{2}} = 1. \tag{4.14}$$

Hence, when $[0, 1]$ is divided into 3 subintervals, the smaller the length of the interval $\frac{1}{k}$ removed, the closer the Hausdorff dimension gets to 1.

In method D, $\dim(D_k) = \frac{\log 2}{\log k}$. So,

$$\lim_{k\to\infty} \dim(D_k) = \lim_{k\to\infty} \left(\frac{\log 2}{\log k}\right) = 0. \tag{4.15}$$

Hence, when $[0,1]$ is divided into 3 subintervals, the smaller the length of the side intervals $\frac{1}{k}$, that is, the larger the length of the interval $(1-\frac{2}{k})$ removed, the closer the Hausdorff dimension gets to 0.

In method E, $\dim(E_k) = \frac{\log \frac{k+1}{2}}{\log k}$ when k is odd. So,

$$\lim_{k\to\infty} \dim(E_k) = \lim_{k\to\infty} \left(\frac{\log \frac{k+1}{2}}{\log k}\right) = 1. \tag{4.16}$$

Hence, when $[0,1]$ is divided into an odd number of equal length intervals, the larger the number of such intervals, the closer the Hausdorff dimension gets to 1.

Conclusion

The Cantor ternary set and the general Cantor sets are all examples of fractal sets. Their self-similarity allows their Hausdorff dimention to be calculated easily, and each is shown to be non-integer. Many questions for further investigation remain, including expoloring other methods of removal. The ultimate question is whether or not it is possible to find a method of removal that will yield a specific given Hausdorff dimension. Considerable work and exploration would need to be done in order to determine this.

Acknowledgements

This paper is the product of a research project undertaken in Summer 2008 through the Department of Mathematics and Physics at Rockhurst University in Kansas City, Missouri with Dr. Zdeňka Guadarrama, Assistant Professor, and Jeanette Powers, undergraduate student. The project was extended with further research for the MT4960 Mathematics Seminar course in Spring 2009 at Rockhurst. The author and the project were generously supported through the James and Elizabeth Monahan Summer Research Fellowship. The author wishes to thank Mr. and Mrs. Monahan and especially Dr. Guadarrama and Ms. Powers.

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