

BOURBAKI IDEALS: A WORKED EXAMPLE

CARRIE WHITTLE

ABSTRACT. Bourbaki's theorem states that if M is a finitely generated torsion-free module of rank n over a Noetherian integrally closed domain R , then M has a free submodule F of rank $n-1$ such that the quotient M/F is isomorphic to an ideal of R . In this paper we compute two examples of Bourbaki ideals for specific modules over the polynomial ring in two variables with rational coefficients.

Bourbaki ideals are those guaranteed by Bourbaki's theorem, which states that if M is a finitely generated torsion-free module of rank n over a Noetherian integrally closed domain R , then M has a free submodule F of rank $n-1$ such that the quotient M/F is isomorphic to an ideal of R . In this paper we examine two examples of Bourbaki ideals.

Let $R = \mathbb{Q}[x, y]$. Then R is Noetherian, since every ideal of $\mathbb{Q}[x, y]$ is finitely generated. Also, R is a unique factorization domain, so R is an integrally closed domain. We will use R as the ring in both our examples. Let $S = R \setminus \{0\}$, so that $S^{-1}R$ is the field of fractions of R .

We begin with our first example. We will call the module in our first example M_1 and the module in our second example M_2 .

$$\text{Let } M_1 = \left\langle \begin{bmatrix} x^2 \\ xy \end{bmatrix}, \begin{bmatrix} xy \\ y^2 \end{bmatrix}, \begin{bmatrix} xy \\ x^2 \end{bmatrix} \right\rangle. \text{ So } M_1 \subseteq R^2.$$

$$\text{Let } a_1 = \begin{bmatrix} x^2 \\ xy \end{bmatrix}, a_2 = \begin{bmatrix} xy \\ y^2 \end{bmatrix}, \text{ and } a_3 = \begin{bmatrix} xy \\ x^2 \end{bmatrix}.$$

Now M_1 is generated by the three elements a_1 , a_2 , and a_3 , so M_1 is finitely generated. We also need it to be torsion-free. If $r \in R$, $m \in M$, and $rm = 0$ imply that $r = 0$ or $m = 0$, we say that the module M over the ring R is torsion-free. Since any submodule of R^2 is torsion-free, M_1 is torsion-free.

Bourbaki's theorem mentions the rank of the module, so let's determine what the rank of M_1 is. The rank of a module is defined as the maximum number of linearly independent elements in the module. Since M_1 is a submodule of R^2 , and the rank of R^2 is 2, the rank of M_1 must be less than or equal to 2. Since a_2 and a_3 are not $S^{-1}R$ -multiples of each other, they are $S^{-1}R$ -linearly independent, and thus R -linearly independent. Since there are two linearly independent elements, a_2 and a_3 , in M_1 , $\text{rank } M_1 \geq 2$. Therefore $\text{rank } M_1 = 2$.

Now we want to find the free submodule F_1 of M_1 which Bourbaki's Theorem guarantees. Let $F_1 = \langle a_3 \rangle$; we want to see that the rank of F_1 is $2 - 1 = 1$. Since F_1 is a submodule of R^2 , which is torsion-free, F_1 is also torsion-free. This means that $\text{rank } F_1 \geq 1$. But since F_1 is generated by one element, $\text{rank } F_1 \leq 1$. Thus $\text{rank } F_1 = 1$. We also want to see that F_1 is free. A module (or submodule) is free

Date: May 15, 2008.

Key words and phrases. Bourbaki ideal.

if it has a basis; that is, a linearly independent set of elements which generate the whole module. In determining the rank of F_1 , we saw that F_1 has a basis, so it is free.

Finally, let's consider the quotient $N_1 = M_1/F_1$. We use \bar{m} to represent the element $m + F_1$ in N_1 (where m is an element of M_1). Now N_1 is generated by the three cosets \bar{a}_1 , \bar{a}_2 , and \bar{a}_3 , but $\bar{a}_3 = \bar{0}$, so $N_1 = \langle \bar{a}_1, \bar{a}_2 \rangle$.

Next we show that N_1 is torsion-free. Let $n_1 \in N_1 \setminus \{0\}$.

Then $n_1 = r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix}$ for some $r_1, r_2 \in R$. Suppose $rn_1 = 0$ in N_1 .

Then $r \left(r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \right) = r_3 \begin{bmatrix} \overline{xy} \\ \overline{x^2} \end{bmatrix}$ in M_1 for some $r_3 \in R$, so

$$\begin{aligned} rr_1x^2 + rr_2xy &= r_3xy & \text{and} & & rr_1xy + rr_2y^2 &= r_3x^2, \text{ so} \\ rr_1x^2y + rr_2xy^2 &= r_3xy^2 & \text{and} & & rr_1x^2y + rr_2xy^2 &= r_3x^3. \end{aligned}$$

Thus $r_3x(y^2 - x^2) = 0$, so $r_3 = 0$, so

$$r \left(r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

However, this implies that $r = 0$, because $n_1 = r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \neq 0$, $r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \in M_1$, and M_1 is torsion-free. Hence N_1 is torsion-free.

Now consider $S^{-1}N_1$, the module generated by N_1 over the field of fractions of R . Since $\frac{1}{x} \in S^{-1}R$,

$$\frac{1}{x} \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \in S^{-1}N_1.$$

Since $S^{-1}N_1$ is generated by N_1 ,

$$\begin{aligned} S^{-1}N_1 &= \left\langle \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix}, \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \right\rangle \\ &= \left\langle x \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}, y \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right\rangle. \end{aligned}$$

In general a module N over a ring R is an R -submodule of $S^{-1}N$ if and only if N is torsion-free, so since N_1 is torsion-free, N_1 is an R -submodule of $S^{-1}N_1$.

Since $S^{-1}N_1$ and $S^{-1}R$ are both vector spaces of dimension 1 over $S^{-1}R$, they are isomorphic. Since they are isomorphic as vector spaces, they are also isomorphic as $S^{-1}R$ -modules and so as R -modules, where an isomorphism is given by mapping any generator of $S^{-1}N_1$ to any generator of $S^{-1}R$. So let's define $\phi_1 : S^{-1}N_1 \rightarrow S^{-1}R$ by sending $\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ to 1. Then ϕ_1 is an isomorphism since $\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ is a generator of $S^{-1}N_1$ and 1 is a generator of $S^{-1}R$.

Now let's take an element of N_1 and see where ϕ_1 sends it. Let $n_1 \in N_1$.

$$\text{Then } n_1 = r_1 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_2 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \text{ for some } r_1, r_2 \in R.$$

Now $n_1 \in S^{-1}N_1$ (since N_1 is an R -submodule of $S^{-1}N_1$), so

$$n_1 = r_1 x \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} + r_2 y \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}.$$

$$\begin{aligned} \text{So } \phi_1(n_1) &= \phi \left(r_1 x \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} + r_2 y \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right) \\ &= r_1 x \phi_1 \left(\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right) + r_2 y \phi_1 \left(\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right) \\ &= r_1 x (1) + r_2 y (1) \\ &= r_1 x + r_2 y. \end{aligned}$$

So $\phi_1(N_1) \subseteq (x, y) \subseteq R$. Hence ϕ_1 sends n_1 into the ideal (x, y) of R . It turns out that every element of (x, y) is the image of an element of N_1 , as we shall see. Let $r \in (x, y)$.

Then $r = r_3 x + r_4 y$ for some $r_3, r_4 \in R$.

$$\text{Now } r_3 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_4 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \in N_1, \text{ and}$$

$$\phi_1 \left(r_3 \begin{bmatrix} \overline{x^2} \\ \overline{xy} \end{bmatrix} + r_4 \begin{bmatrix} \overline{xy} \\ \overline{y^2} \end{bmatrix} \right) = r_3 x + r_4 y = r, \text{ so}$$

$$r \in \phi_1(N_1), \text{ so } (x, y) \subseteq \phi_1(N_1).$$

Thus $\phi(N_1) = (x, y)$. So $N_1 = M_1/F_1 \cong (x, y)$.

Therefore a Bourbaki ideal for M_1 is (x, y) .

Now let us turn to our second example.

$$\text{Let } M_2 = \left\langle \begin{bmatrix} \overline{x^2} \\ \overline{y^2} \\ \overline{x^2} \end{bmatrix}, \begin{bmatrix} \overline{x^2} \\ \overline{x^2} \\ \overline{y^2} \end{bmatrix}, \begin{bmatrix} \overline{y^2} \\ \overline{x^2} \\ \overline{x^2} \end{bmatrix}, \begin{bmatrix} \overline{y^2} \\ \overline{x^2} \\ \overline{y^2} \end{bmatrix}, \begin{bmatrix} \overline{y^2} \\ \overline{y^2} \\ \overline{x^2} \end{bmatrix}, \begin{bmatrix} \overline{x^2} \\ \overline{y^2} \\ \overline{y^2} \end{bmatrix} \right\rangle. \text{ So } M_2 \subseteq R^3.$$

$$\text{Let } b_1 = \begin{bmatrix} \overline{x^2} \\ \overline{y^2} \\ \overline{x^2} \end{bmatrix}, b_2 = \begin{bmatrix} \overline{x^2} \\ \overline{x^2} \\ \overline{y^2} \end{bmatrix}, b_3 = \begin{bmatrix} \overline{y^2} \\ \overline{x^2} \\ \overline{x^2} \end{bmatrix}, b_4 = \begin{bmatrix} \overline{y^2} \\ \overline{x^2} \\ \overline{y^2} \end{bmatrix}, b_5 = \begin{bmatrix} \overline{y^2} \\ \overline{y^2} \\ \overline{x^2} \end{bmatrix}, \text{ and } b_6 = \begin{bmatrix} \overline{x^2} \\ \overline{y^2} \\ \overline{y^2} \end{bmatrix}.$$

Now M_2 is generated by the six elements $b_1, b_2, b_3, b_4, b_5,$ and b_6 , so M_2 is finitely generated. Since any submodule of R^3 is torsion-free, M_2 is torsion-free. We now determine the rank of M_2 . Since M_2 is a submodule of R^3 , whose rank is 3, the rank of M_2 is less than or equal to 3. By row reducing the matrix with columns $b_1, b_5,$ and b_6 , we see that this matrix has rank 3, so its columns are $S^{-1}R$ -linearly independent. Thus $b_1, b_5,$ and b_6 are R -linearly independent, so $\text{rank } M_2 \geq 3$. Hence $\text{rank } M_2 = 3$.

Let $F_2 = \langle b_5, b_6 \rangle$. We want to see that the rank of F_2 is $3 - 1 = 2$. Since b_5 and b_6 are linearly independent, $\text{rank } F_2 \geq 2$. But since F_2 is generated by two elements, $\text{rank } F_2 \leq 2$. Thus $\text{rank } F_2 = 2$. So like F_1 , F_2 has a basis and so is free.

Let $N_2 = M_2/F_2$. Then N_2 is generated by $\overline{b_1}, \overline{b_2}, \overline{b_3}, \overline{b_4}, \overline{b_5}$, and $\overline{b_6}$. But in N_2 , $\overline{b_5} = \overline{0} = \overline{b_6}$, so $N_2 = \langle \overline{b_1}, \overline{b_2}, \overline{b_3}, \overline{b_4} \rangle$.

We used Macaulay 2 to show that N_2 is torsion-free. There is no command in Macaulay such as “IsTorsionFree” to tell the user if a module is torsion-free, so we instead used the command “IsInjective” to see that a particular map is injective, which tells us that the module is torsion-free.

First note that N_2 has rank 1, so we used one torsion-free element - in this case b_1 - to create a free submodule of N_2 ; call it P . Since any free module of rank n is isomorphic to R^n , $P \cong R^1$. So there is an injective R -module homomorphism, say f , from R into N_2 . We defined this map f in Macaulay 2 to be the map sending the standard basis element of R to b_1 . Since, in general, if an R -module homomorphism f from M to N is injective, then the induced module homomorphism $S^{-1}f$ from $S^{-1}M$ to $S^{-1}N$ is injective, we get an induced injective homomorphism f' from $S^{-1}R$ to $S^{-1}N_2$. Now because $S^{-1}R$ and $S^{-1}N_2$ are both vector spaces of dimension 1, they are isomorphic. Since f' is injective, it must be one such isomorphism, so let g be the inverse of f' . Then g maps $S^{-1}N_2$ to $S^{-1}R$. We also have a map, say ψ , from N_2 to $S^{-1}N_2$. We view ψ as being surjective onto its image $\psi(N_2)$. Now ψ is an injective R -module homomorphism if and only if N_2 is torsion-free. And the image of N_2 under $g \circ \psi$ is a finitely generated R -submodule of $S^{-1}R$. Since $S^{-1}R$ is a torsion-free module, multiplication by a nonzero element is an injective R -module homomorphism. We multiplied $g \circ \psi(N_2)$ by the least common denominator of the generating elements of $g \circ \psi(N_2)$ in order to map back into R ; call this multiplication map h . Finally, we used the “IsInjective” command in Macaulay 2 to see that $h \circ g \circ \psi$ is, in fact, injective. Hence ψ must be injective. Thus N_2 is torsion-free.

$$\begin{array}{ccc}
 S^{-1}R & \xleftarrow{g} & S^{-1}N_2 \\
 \uparrow \subseteq & & \uparrow \subseteq \\
 (g \circ \psi)(N_2) & \xleftarrow{g|_{\psi(N_2)}} & \psi(N_2) \\
 \downarrow h & & \uparrow \psi \\
 R & \xrightarrow{f} & N_2
 \end{array}$$

The relationships in $S^{-1}N_2$ are a bit more complicated than they were in $S^{-1}N_1$. Let's begin by looking at the relationship between b_1 and b_2 .

$$\begin{aligned}
 \text{Now } 2b_1 + b_2 &= 2 \begin{bmatrix} x^2 \\ y^2 \\ x^2 \end{bmatrix} + \begin{bmatrix} x^2 \\ x^2 \\ y^2 \end{bmatrix} \\
 &= \begin{bmatrix} 3x^2 \\ x^2 + 2y^2 \\ 2x^2 + y^2 \end{bmatrix} \\
 &= \begin{bmatrix} y^2 \\ y^2 \\ x^2 \end{bmatrix} + 2 \begin{bmatrix} x^2 \\ y^2 \\ y^2 \end{bmatrix} \\
 &= b_5 + 2b_6 \in F_2.
 \end{aligned}$$

So in $S^{-1}N_2$ we have

$$\begin{aligned} 2\bar{b}_1 + \bar{b}_2 &= \bar{0}, \text{ so} \\ \bar{b}_2 &= -2\bar{b}_1. \end{aligned}$$

Next we consider how b_1 and b_3 are related.

$$\begin{aligned} \text{We have } (x^2 + y^2)b_1 + y^2b_3 &= (x^2 + y^2) \begin{bmatrix} x^2 \\ y^2 \\ x^2 \end{bmatrix} + y^2 \begin{bmatrix} y^2 \\ x^2 \\ x^2 \end{bmatrix} \\ &= \begin{bmatrix} x^4 + x^2y^2 + y^4 \\ 2x^2y^2 + y^4 \\ x^4 + 2x^2y^2 \end{bmatrix} \\ &= (x^2 + y^2) \begin{bmatrix} y^2 \\ y^2 \\ x^2 \end{bmatrix} + x^2 \begin{bmatrix} y^2 \\ x^2 \\ y^2 \end{bmatrix} \\ &= (x^2 + y^2)b_5 + x^2b_6 \in F_2. \end{aligned}$$

So in $S^{-1}N_2$,

$$\begin{aligned} (x^2 + y^2)\bar{b}_1 + y^2\bar{b}_3 &= \bar{0}, \text{ and thus} \\ \bar{b}_3 &= \left(\frac{x^2 + y^2}{y^2}\right)\bar{b}_1. \end{aligned}$$

Finally we consider the relationship between b_1 and b_4 .

$$\begin{aligned} \text{Now } (x^2 + 2y^2)b_1 + y^2b_4 &= (x^2 + 2y^2) \begin{bmatrix} x^2 \\ y^2 \\ x^2 \end{bmatrix} + y^2 \begin{bmatrix} y^2 \\ x^2 \\ y^2 \end{bmatrix} \\ &= \begin{bmatrix} x^4 + 2x^2y^2 + y^4 \\ 2x^2y^2 + 2y^4 \\ x^4 + 2x^2y^2 + y^4 \end{bmatrix} \\ &= (x^2 + y^2) \begin{bmatrix} y^2 \\ y^2 \\ x^2 \end{bmatrix} + (x^2 + y^2) \begin{bmatrix} x^2 \\ y^2 \\ y^2 \end{bmatrix} \\ &= (x^2 + y^2)b_5 + (x^2 + y^2)b_6 \in F_2. \end{aligned}$$

So in $S^{-1}N_2$ we have

$$\begin{aligned} (x^2 + 2y^2)\bar{b}_1 + y^2\bar{b}_4 &= \bar{0}, \text{ and so} \\ \bar{b}_4 &= \left(\frac{x^2 + 2y^2}{y^2}\right)\bar{b}_1. \end{aligned}$$

Since $S^{-1}N_2$ is generated by N_2 ,

$$\begin{aligned} S^{-1}N_2 &= \langle \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4 \rangle \\ &= \left\langle \bar{b}_1, -2\bar{b}_1, \left(\frac{x^2 + y^2}{y^2}\right)\bar{b}_1, \left(\frac{x^2 + 2y^2}{y^2}\right)\bar{b}_1 \right\rangle \\ &= \langle \bar{b}_1 \rangle. \end{aligned}$$

Since N_2 is torsion-free, N_2 is an R -submodule of $S^{-1}N_2$.

As with $S^{-1}N_1$, $S^{-1}N_2 \cong S^{-1}R$, where the map sending any generator of $S^{-1}N_2$ to any generator of $S^{-1}R$ is one such isomorphism. So let's define $\phi_2 : S^{-1}N_2 \rightarrow S^{-1}R$ by sending \bar{b}_1 to y^2 . Then ϕ_2 is an isomorphism since \bar{b}_1 is a generator of $S^{-1}N_2$, and y^2 is a generator of $S^{-1}R$. We chose y^2 rather than 1 as the generator of R to use in this isomorphism because we want it to send elements of N_2 to elements of R (not to just any elements of $S^{-1}R$), so we need to get rid of the denominators.

Now let's take an element of N_2 and see where ϕ_2 sends it. Let $n_2 \in N_2$. Then $n_2 \in S^{-1}N_2$ (since N_2 is an R -submodule of $S^{-1}N_2$),

$$\text{so } n_2 = r_1\bar{b}_1 - 2r_2\bar{b}_1 + \left(\frac{x^2 + y^2}{y^2}\right)\bar{b}_1 + \left(\frac{x^2 + 2y^2}{y^2}\right)\bar{b}_1 \text{ for some } r_1, r_2, r_3, r_4 \in R.$$

$$\begin{aligned} \text{So } \phi_2(n_2) &= \phi_2\left(r_1\bar{b}_1 - 2r_2\bar{b}_1 + \left(\frac{x^2 + y^2}{y^2}\right)\bar{b}_1 + \left(\frac{x^2 + 2y^2}{y^2}\right)\bar{b}_1\right) \\ &= r_1\phi_2(\bar{b}_1) - 2r_2\phi_2(\bar{b}_1) + \left(\frac{x^2 + y^2}{y^2}\right)\phi_2(\bar{b}_1) + \left(\frac{x^2 + 2y^2}{y^2}\right)\phi_2(\bar{b}_1) \\ &= r_1y^2 - 2r_2y^2 + r_3(x^2 + y^2) + r_4(x^2 + 2y^2). \end{aligned}$$

$$\text{Hence } \phi_2(N_2) \subseteq (y^2, -2y^2, x^2 + y^2, x^2 + 2y^2) \subseteq R.$$

$$\text{But } (y^2, -2y^2, x^2 + y^2, x^2 + 2y^2) = (x^2, y^2).$$

Hence ϕ_2 sends n_2 into the ideal (x^2, y^2) of R . But every element of (x^2, y^2) is the image of an element of N_2 , as we shall see. Let $r \in (x^2, y^2)$.

$$\text{Then } r = r_5x^2 + r_6y^2 \text{ for some } r_5, r_6 \in R.$$

$$\text{Now } r_5\bar{b}_3 - r_5\bar{b}_1 + r_6\bar{b}_1 \in N_2,$$

$$\text{and } \phi_2(r_5\bar{b}_3 - r_5\bar{b}_1 + r_6\bar{b}_1) = r_5(x^2 + y^2) - r_5y^2 + r_6y^2 = r_5x^2 + r_6y^2 = r,$$

$$\text{so } r \in \phi_2(N_2), \text{ so } (x^2, y^2) \subseteq \phi_2(N_2).$$

$$\text{Thus } \phi_2(N_2) = (x^2, y^2). \text{ So } N_2 = M_2/F_2 \cong (x^2, y^2).$$

Therefore a Bourbaki ideal for M_2 is (x^2, y^2) .

In conclusion, a Bourbaki ideal for $\left\langle \begin{bmatrix} x^2 \\ xy \\ xy \\ x^2 \end{bmatrix} \right\rangle$ over $\mathbb{Q}[x, y]$ is (x, y) ,

and a Bourbaki ideal for $\left\langle \begin{bmatrix} x^2 \\ y^2 \\ x^2 \\ x^2 \end{bmatrix}, \begin{bmatrix} x^2 \\ y^2 \\ x^2 \\ y^2 \end{bmatrix}, \begin{bmatrix} y^2 \\ x^2 \\ y^2 \\ x^2 \end{bmatrix}, \begin{bmatrix} y^2 \\ x^2 \\ y^2 \\ x^2 \end{bmatrix}, \begin{bmatrix} x^2 \\ y^2 \\ x^2 \\ y^2 \end{bmatrix} \right\rangle$ over $\mathbb{Q}[x, y]$ is (x^2, y^2) .

Acknowledgment: The author would like to thank Dr. Mark Rogers for his guidance and assistance in preparing this paper and the presentation it accompanied.

REFERENCES

- [DF04] Dummit, D. S. and Foote, R. M. Abstract Algebra. Hoboken, New Jersey: John Wiley and Sons, Inc., 2004.

DEPARTMENT OF MATHEMATICS, MISSOURI STATE UNIVERSITY
E-mail address: CarrieWhittle@missouristate.edu