

# On the Hartogs phenomenon

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**Abstract** This article is accessible to undergraduate students who have completed a Complex Analysis Course. The main purpose is to present the first step in the theory of Several Complex Variables.

## 1 Holomorphic functions of one variable

The theory of one complex variable (see [GK] for complete overview) is significantly different from that of several variables. Nevertheless it is reasonable to present the main ideas and make connections and comparisons with the further material. An analytic function of one variable has the general form:

$$f(z) = \sum_{i=-\infty}^{\infty} a_i(z-b)^i$$

and, by the Root Test, converges for those  $z \in \mathbb{C}$  which fulfill  $r < |z-b| < R$ , where

$$b, a_i \in \mathbb{C}, \quad r = \limsup_{i < 0} \sqrt[i]{|a_i|}, \quad \frac{1}{R} = \limsup_{i > 0} \sqrt[i]{|a_i|} \quad (1)$$

It is particularly worthwhile to mention that  $r$  could be equal to 0 and  $R$  could be  $\infty$ . It might happen that the region of convergence is empty. We will not discuss convergence on the boundary, i.e. for  $|z-b| = r$  or  $|z-b| = R$  which makes a good topic, just for another paper.

Sketching a picture of the annulus of convergence of the function  $f$  of a single variable we use the complex plane. If the center of annulus is  $b = 0$  then it is sufficient to sketch only the magnitude

of the variable  $z$ . This way we can use 1-dimensional picture to represent the regions of convergence of functions of one complex variable.

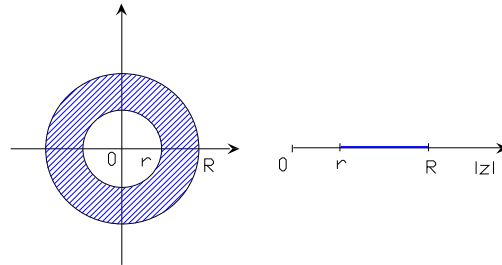


Figure 1: Two sketches of annulus centered at the origin.

## 2 Holomorphic functions of two or more variables

Any analytic function of two variables can be written as a series of the following form:

$$f(z, w) = \sum_{i, j = -\infty}^{\infty} a_{ij} (z - b)^i (w - c)^j$$

with  $(z, w) \in \mathbb{C}^2$  and  $b, c \in \mathbb{C}$ . For further discussion we will assume that  $b = c = 0$ . In the next sections we will build some intuition about possible regions of convergence of holomorphic functions of two or more variables. For complete material refer to [K], [GR] or online sources [KW] and [R]. The pictures which follow the examples, show the regions of convergence with the magnitudes of the complex variables  $z$  and  $w$  on the axes. This way we can do sketches of 4-dimensional objects in 2-dimensions.

## 2.1 Examples

**Example 2.1.1** Starting with the simplest example let us consider a function of two variables which is a product of functions of one variable:

$$f_1(z, w) = g(z)h(w).$$

The function  $f_1$  is holomorphic for those  $(z, w) \in \mathbb{C}^2$  where  $g(z)$  and  $h(w)$  are holomorphic. So in

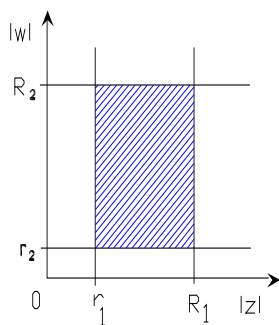


Figure 2: Product of two annuli centered at the origin.

this example the region of convergence is simply a product of annuli:

$$\{(z, w) \in \mathbb{C}^2 : r_1 < |z| < R_1, r_2 < |w| < R_2\},$$

where  $0 \leq r_i \leq R_i \leq \infty$  for  $i = 1, 2$ .

**Example 2.1.2** Consider a holomorphic function of the following form:

$$f_2(z, w) = \sum_{i=-\infty}^{\infty} a_i (zw)^i.$$

Since  $f_2$  depends on the product  $zw$ , we can introduce a new variable  $u = zw$  and claim that  $f_2$  converges for  $(z, w) \in \mathbb{C}^2$  so that  $r < |zw| < R$  for  $r, R \geq 0$  defined as in equation (1).

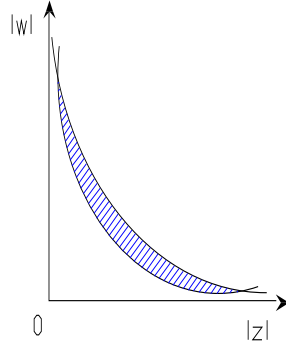


Figure 3: The region of convergence of  $f_2$ .

## 2.2 Main theorems

In this section we mention few theorems which clarify some properties of the regions of convergence of functions of several complex variables. First, a holomorphic function of one variable can have singularities at isolated points, which can not happen for several variables. Generally, the set of singularities can not be “too small” in a sense of dimension:

**Theorem 2.2.1** [Gu], Vol I, part D, Theorem 4 *Suppose that  $E$  is an analytic subset of  $\mathbb{C}^n$ , ( $n \geq 2$ ) of complex dimension at most  $n - 2$ , then every function holomorphic on  $\mathbb{C}^n \setminus E$  can be extended holomorphically to  $\mathbb{C}^n$ .*

Particular, if  $n = 2$  the analytic subsets of dimension  $n - 2 = 0$  are simply points. For example, the function  $f(z) = \frac{1}{z}$  is not holomorphic at the point  $z = 0$  in  $\mathbb{C}^1$ . But in  $\mathbb{C}^2$  the set of singularities for the function  $f(z, w) = \frac{1}{z}$  describes a line. This example builds intuition for the following theorem:

**Theorem 2.2.2** [KK] Corollary 7.11 *Let  $f$  be holomorphic on  $\mathbb{C}^n \setminus E$ , ( $n \geq 2$ ) where  $\dim E = n - 1$ , then  $E$  is a set of zeros of a holomorphic function.*

It is clear that if  $h$  is a holomorphic function then  $f = \frac{1}{h}$  has singularities at those points where  $h$  is equal to 0.

What happens if the set  $E$  has the same dimension as the whole space? Then we can still mention the following result often called the Hartogs phenomenon.

**Theorem 2.2.3** (*[R], section 3, Lemma 2*) *Let  $E$  be a compact set of a domain  $\mathbb{C}^n$ ,  $n \geq 2$ , with connected  $\mathbb{C}^n \setminus E$ . Then every function holomorphic on  $\mathbb{C}^n \setminus E$  can be extended holomorphically to  $\mathbb{C}^n$ .*

The set  $E$  does not need to be compact. The following result, often called the Hartogs figure, seems to be the most surprising, but is crucial in the theory of Several Complex Variables.

**Theorem 2.2.4** (*[KK] E. 11b.*) *Let  $f$  be a holomorphic function on the domain*

$$\{(z, w) \in \mathbb{C}^2 : |z| < 2, |w| < 1\} \cup \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 2\}$$

*then  $f$  has a holomorphic extension to:*

$$\{(z, w) \in \mathbb{C}^2 : |z| < 2, |w| < 2\}.$$

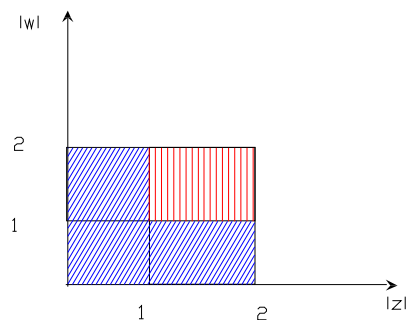


Figure 4: The Hartogs Figure

It can be explained in the following way: having a function holomorphic in a nonconvex set described by the magnitudes of the variables, we can obtain an extension to fill the set to be convex. The theorem presented here is in fact significantly simpler than the general one but made an excellent source of intuition for Hartogs, who proved it in 1906. After this result appeared, it became clear that the theory of Several Complex Variables can not be a natural generalization of the theory of Single Complex Variable.

All theorems mentioned above are proven for  $n \geq 2$ . The theory of Complex Analysis works with more general objects, called complex manifolds which are glued out of copies of  $\mathbb{C}^n$ . Theorems 2.2.1 and 2.2.2 remain true for complex manifolds but Theorem 2.2.3 does not need to be. The results for particular manifolds will be very interesting for the general theory.

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