On the Hartogs-Bochner Phenomenon

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Abstract

This article is accessible to undergraduate students who have completed a complex analysis course. The main purpose is to present the first steps in the theory of several complex variables and the Cauchy-Riemann theory.

1 Holomorphic functions of one variable

In the identification of $\mathbb{C}$ with $\mathbb{R}^2$, a complex number is identified with a pair of real numbers, its real and imaginary part $z = (x, y) = x + iy$.

A continuous function $f(z) = u(x, y) + iv(x, y)$ of one complex variable is called holomorphic ([GK]) if it fulfills the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which can be abbreviated as $\frac{\partial f}{\partial \bar{z}} = 0$ or $\overline{\partial} f = 0$.

2 Holomorphic functions of two or more variables

Similarly, $\mathbb{C}^2$ can be identified with $\mathbb{R}^4$, and we can use the notation:

$$(z_1, z_2) = (x_1, y_1, x_2, y_2) = (x_1 + iy_1, x_2 + iy_2).$$

A continuous function of two complex variables

$$f(z_1, z_2) = u(x_1, y_1, x_2, y_2) + iv(x_1, y_1, x_2, y_2)$$

is called holomorphic (1.1 Definition in [KK]) if it fulfills the Cauchy-Riemann equations with respect to z_1 and z_2:

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial y_1}, \ \frac{\partial u}{\partial y_1} = -\frac{\partial v}{\partial x_1}$$

and

$$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial y_2}, \ \frac{\partial u}{\partial y_2} = -\frac{\partial v}{\partial x_2},$$

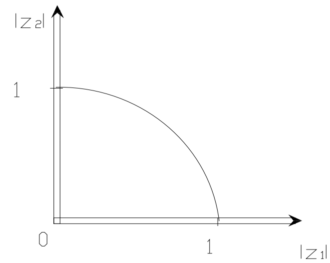

Figure 1: A sketch of the unit sphere in $\mathbb{C}^2$.

which can be abbreviated as $\frac{\partial f}{\partial \overline{z_1}} = 0$ and $\frac{\partial f}{\partial \overline{z_2}} = 0$, respectively, or in an even shorter form as $\overline{\partial}_{z_1} f = 0$ and $\overline{\partial}_{z_2} f = 0$. Figure 1 shows a sketch of the unit ball described by the equation $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$ in $\mathbb{C}^2$. Complex directions are marked by double lines.

Although the definitions appear almost identical, the theory of one complex variable is not comparable with that of several variables. The theory of several variables admits some remarkable properties, one of which is the Hartogs-Bochner phenomenon. Here is one of its versions for $\mathbb{C}^n$ with $n \geq 2$:

Theorem 2.1 *(Chapter I, Section C, 5. Theorem [GR]) Let a function f be holomorphic on a neighborhood of the boundary of a ball in $\mathbb{C}^n$ with $n \geq 2$. Then f can be holomorphically extended to the entire ball.*

Figure 2 shows a sketch of the unit sphere in $\mathbb{C}^1 \times \mathbb{R}^2$. Complex directions are marked by double lines.

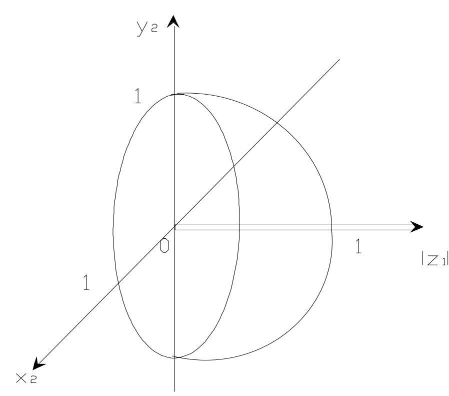

Figure 2: A sketch of the unit sphere in $\mathbb{C}^1 \times \mathbb{R}^2$.

Example 2.1 *This result cannot occur in the theory of one variable. Consider the function $f(z) = \frac{1}{z}$. It is holomorphic on a neighborhood of the unit circle but cannot be extended to the entire unit disc.*

There are more phenomena in $\mathbb{C}^n$ for $n \geq 2$. Here, a domain U is an open, connected, relatively compact set with a smooth connected boundary.

Definition 2.1 *The Hartogs-Bochner phenomenon holds for a domain $U \Subset X$ if any smooth CR function on ∂U can be holomorphically extended to U and smoothly extended up to the boundary.*

Definition 2.2 *The Hartogs-Bochner phenomenon holds in a complex manifold X if it holds for any domain $U \subset X$.*

3 Cauchy-Riemann theory

The Cauchy-Riemann theory works with:

1. the geometry of real objects in complex spaces together with the induced complex structure and
2. the corresponding theory of functions.

The following examples (Part II, Chapter 7, Example 1 [B]) explain this theory more precisely.

Example 3.1 *Let S be the unit sphere in $\mathbb{C}^2 \simeq \mathbb{R}^4$ described by the equation*

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$$

or, in complex notation,

$$|z_1|^2 + |z_2|^2 = 1.$$

Consider the point $p = (1, 0, 0, 0) \in S$. The tangent space at p is spanned by $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial x_2}$ and

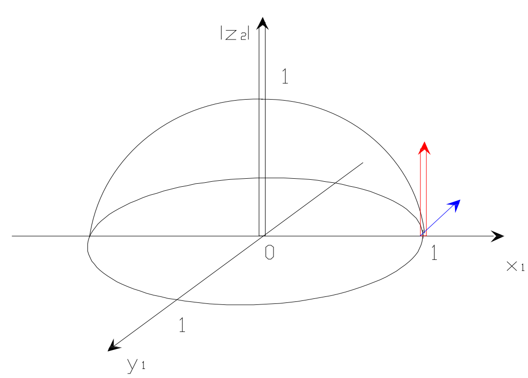

Figure 3: A sketch of the unit sphere in $\mathbb{R}^2 \times \mathbb{C}^1$ with tangent vectors.

$\frac{\partial}{\partial y_2}$. Figure 3 shows a sketch of the unit sphere in $\mathbb{R}^2 \times \mathbb{C}^1$ with tangent vectors. Complex vectors are marked by double lines.

Notice that $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_2}$ span a complex direction tangential to S.

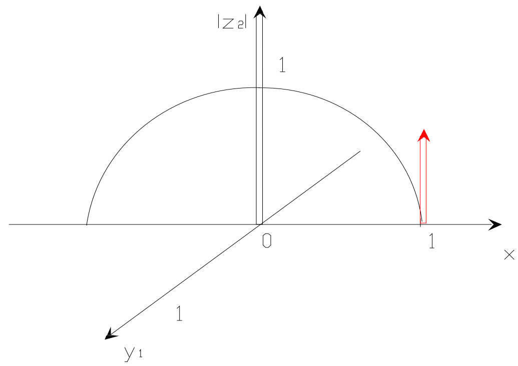

Figure 4: A sketch of a great circle with a complex tangent vector.

Example 3.2 *Let M be a great circle on the unit sphere in $\mathbb{C}^2 \simeq \mathbb{R}^4$ described by the equations*

$$y_1 = 0 \qquad \text{and} \qquad x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$$

or, in complex notation,

$$Imz_1 = 0 \qquad \text{and} \qquad |z_1|^2 + |z_2|^2 = 1.$$

Consider the point $p = (1, 0, 0, 0) \in S$. Then the tangent space at p is spanned by $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_2}$. Notice that $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_2}$ span a complex direction tangential to M. Figure 4 shows a sketch of this great circle with a complex tangent vector.

Now, consider the point $q = (0, 0, 1, 0) \in S$. Tangent space at q is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_2}$. Notice that $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_2}$ DO NOT span a complex direction tangential to M.

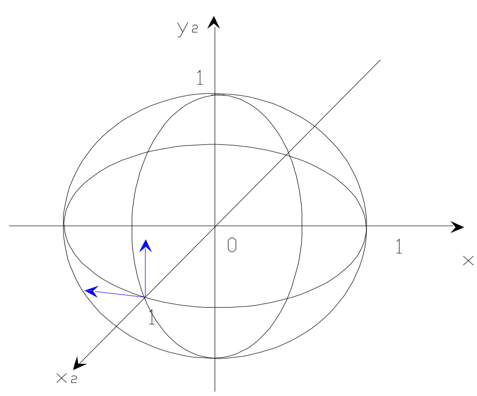

Figure 5: A sketch of a great circle with real tangent vectors.

Those real objects, for which the dimension of the tangential complex space remains equal for all points, are called Cauchy-Riemann objects. A function that fulfills $\overline{\partial}_w f = 0$ for all tangential complex directions w is called a Cauchy-Riemann function.

The existence of an extension of a Cauchy-Riemann function from a Cauchy-Riemann hypersurface makes an interesting object of investigation (Section 14.1 [B]). Extensions for objects of higher codimension are also worth consideration (Section 14.2 [B]).

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