Calculus Bee

Solutions

Spring 2003

1. Evaluate the following limits.

   $$\lim_{x\to 0}\frac{\sin(4x)}{2x}$$

   $$\lim_{x\to 0}\frac{\sin(4x)}{2x}=\lim_{t\to 0}\frac{\sin t}{t/2}\quad (\text{where } t=4x)$$
   $$=2\lim_{t\to 0}\frac{\sin t}{t}=2$$

   $$\lim_{x\to\infty}\sqrt{\frac{18x+7}{8x-5}}$$

   $$\lim_{x\to\infty}\sqrt{\frac{18x+7}{8x-5}}=\sqrt{\lim_{x\to\infty}\frac{18x+7}{8x-5}}=\sqrt{\frac{18}{8}}=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

2. Find the derivatives of the following functions.

   $$y=\ln\left(t^3-2\right)$$

   $$y'=\frac{1}{t^3-2}\frac{d}{dt}\left(t^3-2\right)=\frac{3t^2}{t^3-2}$$

   $$f(x)=e^{\cos x}$$

   $$f'(x)=e^{\cos x}\left(-\sin x\right)$$

3. Find all values of C for which the line $y=15x+C$ will be tangent to the graph of $y=x^3+6x^2-13$.

   $$y=x^3+6x^2-13\Rightarrow y'=3x^2+12x$$
   $$3x^2+12x=15\Rightarrow x=-5,1$$
   $$x=-5\Rightarrow y=(-5)^3+6(-5)^2-13=12\Rightarrow C=12-15(-5)=87$$
   $$x=1\Rightarrow y=(1)^3+6(1)^2-13=-6\Rightarrow C=-6-15(1)=-21$$

   So C = 87 or –21

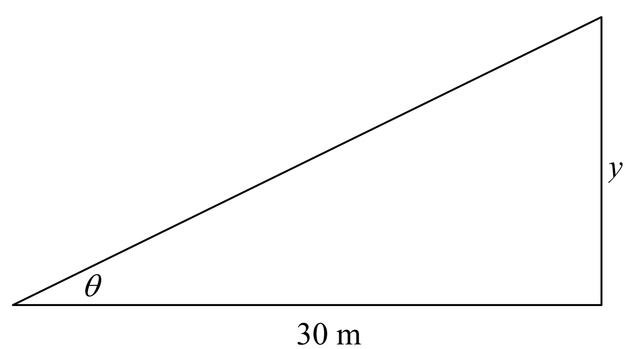
4. Find **all** of the points (x, y) on the curve $x^2 + 4x^2y + y^2 = 16$ at which the curve has a horizontal tangent line.

$$x^2 + 4x^2y + y^2 = 16 \Rightarrow 2x + 8xy + 4x^2y' + 2yy' = 0$$
$$\Rightarrow 2x + 8xy = -4x^2y' - 2yy'$$
$$\Rightarrow 2x + 8xy = y'\left(-4x^2 - 2y\right)$$
$$\Rightarrow y' = \frac{2x + 8xy}{-4x^2 - 2y}$$

$$y' = 0 \Rightarrow \frac{2x + 8xy}{-4x^2 - 2y} = 0 \Rightarrow 2x + 8xy = 0 \Rightarrow x = 0 \text{ or } y = -1/4$$
$$x = 0: x^2 + 4x^2y + y^2 = 16 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$
$$y = -1/4: x^2 + 4x^2y + y^2 = 16 \Rightarrow 1/16 = 16 \Rightarrow \text{contradiction}$$

two points: $(0, -4)$ and $(0, 4)$

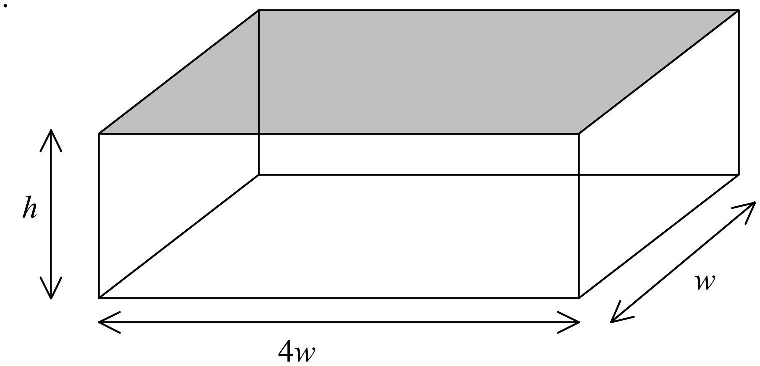
5. A balloon is rising at a rate of 3 meters per second from a point on the (level) ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.



Find $\frac{d\theta}{dt}$ when $\frac{dy}{dt} = 3, y = 30$

$$y = 30\tan\theta \Rightarrow \frac{dy}{dt} = 30\left(\sec^2\theta\right)\frac{d\theta}{dt}$$
$$\Rightarrow 3 = 30\left(\sec^2\frac{\pi}{4}\right)\frac{d\theta}{dt} = 30(2)\frac{d\theta}{dt}$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{3}{60} = \frac{1}{20} \text{ radians per second}$$

6. Paul wants to make a box to hold 1200 cubic feet of marbles. The bottom and the sides of the box will be made out of oak, which will cost \$1 per square foot. The top, however, will be made of elacca wood, which costs \$2 per square foot. To make matters slightly more difficult, Paul only has 64 square feet of elacca wood total. Given that the length of the box must be four times the width (w), while the height (h) has no such restriction, find the cost of the cheapest box Paul can make.



Goal: minimize $1\left[4w^2+2(4wh)+2(wh)\right]+2\left[4w^2\right]$ subject to $4w^2h=1200$

Due to the constraint on elacca wood, we have to have $4w^2 \le 64$, so our interval for w is $0<w\le 4$.

$$4w^2h=1200 \Rightarrow h=\frac{300}{w^2}$$

$$1\left[4w^2+2(4wh)+2(wh)\right]+2\left[4w^2\right]=12w^2+10wh$$

$$=12w^2+10w\left(\frac{300}{w^2}\right)$$

$$=12w^2+\frac{3000}{w} \xleftarrow{\text{call this } C(w)}$$

$$C(w)=12w^2+\frac{3000}{w} \Rightarrow C'(w)=24w-\frac{3000}{w^2}$$

$$24w-\frac{3000}{w^2}=0 \Rightarrow 24w=\frac{3000}{w^2} \Rightarrow w^3=\frac{3000}{24}=125 \Rightarrow w=5$$

Since we have to have $0<w\le 4$, the minimum must occur at the endpoint $w=4$

$$C(4)=12(4)^2+\frac{3000}{4}=\$\,942$$

7. Let

$$f(x)=\begin{cases}\cos x & \text{if } x\le \pi/2,\\ ax^2+b & \text{if } x>\pi/2.\end{cases}$$

Describe all values of *a* and *b* for which *f* is differentiable for all real numbers *x*.

$$f'(x)=\begin{cases}-\sin x & \text{if } x<\pi/2,\\ 2ax & \text{if } x>\pi/2\end{cases}$$

For *f* to be differentiable at $\pi/2$, we need $2a\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}$, i.e. $a=-\frac{1}{\pi}$.

However, we also need *f* to be continuous at $\pi/2$, so we need

$$\cos\frac{\pi}{2}=a\left(\frac{\pi}{2}\right)^2+b=-\frac{1}{\pi}\left(\frac{\pi}{2}\right)^2+b,\text{ so } b=\frac{\pi}{4}.$$

$$a=-\frac{1}{\pi}\quad b=\frac{\pi}{4}$$

8. Suppose that $f(x)$ is continuous on the closed interval $[a,b]$, and there exists real numbers *c* and *d* with $a<c<d<b$ such that $f(x)>0$ only on the open interval (c,d). If

$$\int_a^d f(x)\,dx=12,\quad \int_c^b f(x)\,dx=9,\text{ and } \int_d^b f(x)\,dx=2\int_a^c f(x)\,dx,$$

find $\int_a^b f(x)\,dx.$

Let $A=\int_c^d f(x)\,dx.$

$$\int_a^d f(x)\,dx=12\Rightarrow \int_a^c f(x)\,dx=12-A$$

$$\int_c^b f(x)\,dx=9\Rightarrow \int_d^b f(x)\,dx=9-A$$

$$\int_d^b f(x)\,dx=2\int_a^c f(x)\,dx\Rightarrow 9-A=2(12-A)\Rightarrow A=15$$

So

$$\begin{aligned}\int_a^b f(x)\,dx&=\int_a^c f(x)\,dx+\int_c^d f(x)\,dx+\int_d^b f(x)\,dx\\ &=(12-15)+15+(9-15)\\ &=6\end{aligned}$$

9. Find the particular solution to the differential equation $yy' - e^x = 0$ which satisfies the initial condition $y(0) = 4$.

$$
\begin{aligned}
yy' - e^x = 0 &\Rightarrow yy' = e^x \\
&\Rightarrow y\frac{dy}{dx} = e^x \\
&\Rightarrow ydy = e^x dx \\
&\Rightarrow \int y\,dy = \int e^x dx \\
&\Rightarrow \frac{y^2}{2} = e^x + C
\end{aligned}
$$

$$y(0) = 4 \Rightarrow \frac{4^2}{2} = e^0 + C \Rightarrow C = 7$$

$$
\begin{aligned}
\frac{y^2}{2} = e^x + C = e^x + 7 &\Rightarrow y^2 = 2e^x + 14 \\
&\Rightarrow y = \sqrt{2e^x + 14}
\end{aligned}
$$

10. Evaluate the following indefinite integral.

$$\int \frac{e^x \cos(e^x)}{1 + \sin^2(e^x)}\,dx$$

Let $u = \sin(e^x)$

$du = \cos(e^x)e^x dx$

$$
\begin{aligned}
\int \frac{e^x \cos(e^x)}{1 + \sin^2(e^x)}\,dx &= \int \frac{du}{1 + u^2} \\
&= \arctan u + C \\
&= \arctan(\sin(e^x)) + C
\end{aligned}
$$