

Working with Circles

Definition of a circle: A circle is the set of all points that are the same distance from a fixed point called the *center*. The distance from the center to any point on the circle is called the *radius*.

Topics covered in this handout:

1. Standard form of the equation of a circle
2. Completing the square for variables in the equation of a circle
3. Finding the center and radius of a circle
4. Graphing a circle

Standard Form of a Circle:

The standard form of the equation of a circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

Notice the negative signs in front of the h and the k . You must take this into account when writing the equation of a circle.

Example: Let $(h, k) = (4, -7)$ and $r = 3$. Write the equation of this circle in standard form.

Plugging into the equation 4 for h , -7 for k , and 3 for r you get

$$(x - (4))^2 + (y - (-7))^2 = 3^2$$

$$(x - 4)^2 + (y + 7)^2 = 9$$

Completing the Square:

The equation of a circle is not always given to you in standard form. So, you have to use the method of completing the square to rewrite the equation into standard form.

First, let's look at completing the square for any type of equation that has one variable.

Example: $x^2 + 8x + 14 = 0$

First, subtract 14 from both sides to get only terms with an x or x^2 on one side.

$$x^2 + 8x = -14$$

You need to find a number that you can add to the left-hand side of the equation so it will factor as a perfect square.

$$x^2 + 8x + \underline{\quad} = -14$$

To find this, divide the number in front of the x by 2 (i.e. $8/2 = 4$). Then, square the result you just obtained (i.e. $(4)^2 = 16$). Add this number to the left-hand side of the equation. You must also add this number to the right-hand side of the equation so that you don't change the original equation (Remember in algebra, if you do it to one side of the equation, you must do it to the other side).

$$x^2 + 8x + \underline{16} = -14 + \underline{16}$$

$$(x + 4)(x + 4) = 2$$

$$(x + 4)^2 = 2 \quad [\text{Hint: You computed } 8/2 = 4 \text{ above, and the perfect square ended up being } (x + 4)^2. \text{ This will always happen when completing the square.}]$$

Example: Let's now complete the square for both variables in the equation of a circle.

$$x^2 - 4x + y^2 + 6y - 3 = 0$$

$$x^2 - 4x + y^2 + 6y = 3 \quad [\text{Terms with a variable on one side, constants on the other side}]$$

$$(x^2 - 4x + \underline{\quad}) + (y^2 + 6y + \underline{\quad}) = 3 \quad [-4/2 = -2, (-2)^2 = 4 ; 6/2 = 3, (3)^2 = 9]$$

$$(x^2 - 4x + \underline{4}) + (y^2 + 6y + \underline{9}) = 3 + \underline{4} + \underline{9}$$

$$(x - 2)(x - 2) + (y + 3)(y + 3) = 16$$

$$(x - 2)^2 + (y + 3)^2 = 16 \quad [\text{Remember the hint from above } -4/2 = -2 \text{ and } 6/2 = 3]$$

Finding the Center and Radius of a Circle:

Find the center and radius of a circle that is written in standard form.

Example: $(x - 5)^2 + (y + 6)^2 = 49$

[Remember, the standard form of the equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$ where (h, k) is the center and r is the radius.]

Center: (h, k) = (5, -6) Radius: $r = \sqrt{49} = 7$

Example: $x^2 + (y - 9)^2 = 29$ [Think of it as $(x - 0)^2 + (y - 9)^2 = 29$]

Center: (0, 9) Radius: $r = \sqrt{29}$

Example: Find the center and radius of equation that is not given in standard form.

$$x^2 + 2x + y^2 + 4y - 5 = 0$$

$$(x^2 + 2x + \underline{\quad}) + (y^2 + 4y + \underline{\quad}) = 5$$

$$(x^2 + 2x + \underline{1}) + (y^2 + 4y + \underline{4}) = 5 + \underline{1} + \underline{4} \quad \text{[Complete the square]}$$

$$(x + 1)^2 + (y + 2)^2 = 10$$

Center: (-1, -2) Radius: $r = \sqrt{10}$

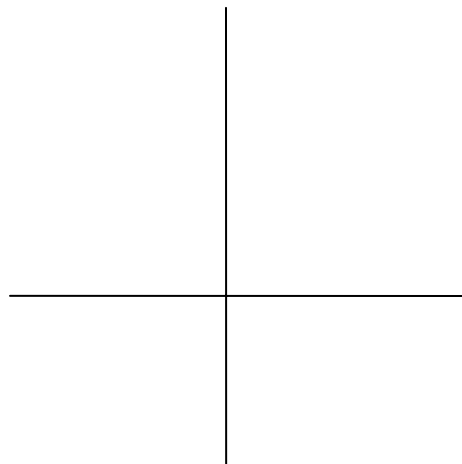
Graphing a Circle:

If the equation is in standard form, you can easily obtain the information needed to graph the circle. If it is not in standard form, complete the square so you can then write it in standard form.

Example: $(x - 2)^2 + (y - 3)^2 = 16$

You know the center of this circle is (2, 3), so place that point on your graph first.

You also know that the radius is $\sqrt{16} = 4$. Remember the radius is the distance from the center to any point on the circle. So, moving 4 units to the right from the center will give you a point on the circle (6, 3). Likewise, you can move 4 units to the left from the center to obtain (-2, 3), 4 units up from the center to obtain (2, 7), and 4 units down from the center to obtain (2, -1). You now have four points on our circle along with the center to sketch a graph.



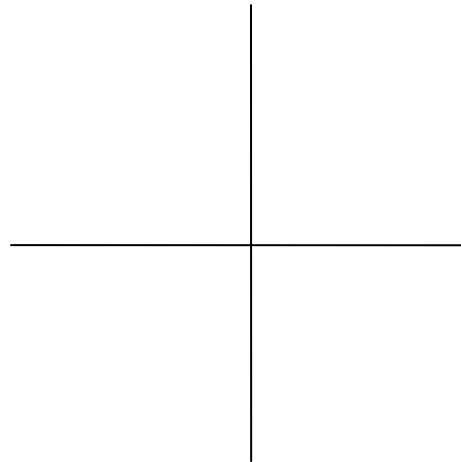
Example: Graphing a circle whose radius is not an integer.

$$(x + 1)^2 + (y - 3)^2 = 10$$

Center: $(-1, 3)$ Radius: $\sqrt{10}$

Even if the radius is not an integer, you still follow the same process to obtain points on the circle. Move $\sqrt{10}$ units to the right from the center to obtain the point $(-1 + \sqrt{10}, 3)$, $\sqrt{10}$ units to the left from the center to obtain $(-1 - \sqrt{10}, 3)$, $\sqrt{10}$ units up from the center to obtain $(-1, 3 + \sqrt{10})$, $\sqrt{10}$ units down from the center to obtain $(-1, 3 - \sqrt{10})$.

Since $\sqrt{9} = 3$, you know that $\sqrt{10}$ is slightly bigger than 3 [$\sqrt{10} \approx 3.16$]. This will help you approximate the points when sketching the graph of the circle. For example, the point $(-1 + \sqrt{10}, 3)$ is approximately the point $(-1 + 3.16, 3) = (2.16, 3)$.



Try on your own: 1) Find the center and radius of this circle. (Answers at bottom of page)

$$(x - 8)^2 + (y + 5)^2 = 15$$

Try on your own: 2) Complete the square to get the equation into standard form.

$$x^2 + 10x + y^2 - 4y + 20 = 0$$

1) Center: $(8, -5)$ Radius: $\sqrt{15}$

2) $(x + 5)^2 + (y - 2)^2 = 9$