Finding Domain and Asymptotes of Rational Functions

A **rational function** is a function that is a fraction of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x)$ does not equal zero.

Some examples of rational functions are as follows:

$$f(x) = \frac{x-1}{x^2 - x - 2} \qquad g(x) = \frac{1}{x+2} \qquad h(x) = \frac{x}{x^2+1}$$

where f, g, and h are the names of the functions and x is the independent variable.

I. <u>Finding Domain</u>

In general, the domain of a rational function of x includes all real numbers except x-values that make the denominator zero.

1. Determine whether or not the function has a denominator with an independent variable, namely x.
2. If so, set the denominator equal to zero and solve for x. Exclude any solutions that are real numbers from the domain.
3. If the function has no variable in the denominator or the solutions of the denominator when set equal to zero are not real numbers, the domain of the function is all real numbers.

Example A

$$g(x) = \frac{1}{x+2}$$

Step 1: The function has a variable in the denominator, namely x.

Step 2: Set the denominator equal to zero.

$x + 2 = 0$
$x = -2$

Step 3: The domain of function g is all real numbers except $x = -2$.

Example B

$$f(x) = \frac{x-1}{x^2 - x - 2}$$

Step 1: The function has a variable in the denominator, namely x.

Step 2: Set denominator equal to zero.

$$x^2 - x - 2 = 0$$

Factor $\quad (x+1)(x-2) = 0$

Solve $\quad x = -1 \qquad x = 2$

Step 3: The domain of f is all real numbers except $x = -1$ and $x = 2$.

Example C

$$h(x) = \frac{x}{x^2 + 1}$$

Step 1: The function has a variable in the denominator, namely x.

Step 2: Set the denominator equal to zero.

$$x^2 + 1 = 0$$

Solve $\quad x^2 = -1$

$x = \pm\sqrt{-1}$ (no real solutions)

Step 3: The domain of h is all real numbers.

Example D

$$k(x) = \frac{x^2 + 4x + 4}{3}$$

Step 1: The function has no variable in the denominator.

Step 2: The domain of k is all real numbers.

II. Finding Vertical Asymptotes

Vertical asymptotes of a function are vertical lines that the graph of the function is always approaching but never touching. They serve as boundaries of the function's graph. Vertical asymptotes are found much in the same manner as the domain, except they take on the values of the independent variable *excluded* from the function's domain.

1. Set the denominator equal to zero and solve.

Example $\quad \dfrac{1}{x-4}$

Step 1: $x-4=0$

The line $x=4$ is the vertical asymptote.

III. **Finding Horizontal Asymptotes**

Horizontal asymptotes of a function are horizontal lines that the graph of the function is always approaching but never touching. Like vertical asymptotes, they also serve as the boundaries of the function's graph. Whether or not a graph has a horizontal asymptote is based on the degrees of the independent variables in the numerator and denominator.

1. Look at the highest degree of the independent variable in both the numerator and the denominator.
2. There are three possibilities:
   a) If the numerator has lower degree than the denominator, the horizontal asymptote is $y=0$, the x-axis.

   b) If the numerator and denominator have the same degree, write the numerator and denominator in descending order, i.e.
   $$f(x)=\frac{a_n x_n+\ldots+a_0}{b_n x_n+\ldots+b_0}.$$
   Then the horizontal asymptote is the line $y=\frac{a_n}{b_n}$.

   c) If the degree of the numerator is higher than the degree of the denominator, then the graph of f has no horizontal asymptote.

Example A

Step 1: $f(x)=\dfrac{x-1}{x^2+2}$

Step 2: The degree of the variable in the numerator is less than the degree of the variable in the denominator. The line $y=0$ is the horizontal asymptote.

Example B

Step 1: $f(x)=\dfrac{3x^2+x+1}{4x^2+2}$

Step 2: The numerator and denominator have the same degree. The line $y=\frac{3}{4}$ is the horizontal asymptote.

Example C

Step 1: $f(x) = \frac{x^2 + 2}{x}$

Step 2: The degree of the numerator is higher than the degree of the denominator. There is no horizontal asymptote.

IV. Finding Slant or Oblique Asymptotes

Slant (also called oblique) asymptotes of a function are straight, diagonal lines that the graph of the function is always approaching but never touching. They also serve as boundaries of the function's graph. These asymptotes exist *if and only if* the degree of the numerator is exactly one greater than the degree of the denominator.

1. Look to see that the degree of the numerator is one greater than the degree of the denominator. If not, there is no slant asymptote.
2. Divide the numerator by the denominator using long division and drop the remainder. The result is the slant asymptote.

Example $f(x) = \frac{x^3}{x^2 + 1}$

Step 1: $\frac{x^3}{x^2 + 1}$. The degree of the numerator is 3, which is greater than the degree of the denominator by exactly one, since the degree of the denominator is 2.

Step 2: When we divide x^3 by $x^2 + 1$ using long division, the result is $x - \frac{x}{x^2 + 1}$. When we drop the remainder, the slant asymptote is the line $y = x$.